PERGAMON

International Journal of Solids and Structures 37 (2000) 4825–4840

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

K-Dominance zone for a semi-infinite mode I crack in a sandwich composite

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Received 22 October 1998; in revised form 18 May 1999

Abstract

A Mode I semi-infinite crack located at the midplane of an elastic layer sandwiched between two identical half-spaces is considered. The complete stress distribution for the eigenproblem with traction-free crack faces and loading at infinity is obtained. The auxiliary problem for a loaded crack is first solved using the Wiener–Hopf method and an eigensolution is derived by a limiting procedure. The solution is presented in closed form in terms of double quadratures. Based on an analysis of the stress field the size of the K-dominance zone is determined and restrictions are established on employing the conventional K-concept. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Sandwich composite; Eigen solution; K-Dominance zone; Winer–Hope method

1. Introduction

Fracture analysis of multilayered bodies incorporating the stress intensity factor approach is based on the assumption that the size of the inelastic process zone R_{p} near the crack tip is small compared to the size of the K-dominance zone R_K, defined by the crack length, distance from the crack to the interface and loading length parameter; i.e.

$$R_{\mathrm{p}} \ll R_K. \tag{1}$$

Only in this case does the stress-intensity factor approach correspond to the energy release rate

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PII: S0020-7683(99)00193-6

Nomenclature

R_p	size of the inelastic fracture process zone
R_K	size of the stress intensity factor dominance zone
j	index denoting the layer ($j = 1$) or the half-plane ($j = 2$) related quantities
μ_j	shear modulus
ν_j	Poisson ratio
$u^{(j)}$, $v^{(j)}$	displacements in the x and y directions respectively in the auxiliary problem
$\sigma_y^{(j)}$, $\tau_{xy}^{(j)}$	stress field components in the auxiliary problem
$q(x)$	loading on the crack faces
λ	real parameter of the Fourier transform with respect to x
$A_i^{(j)}$	functions of λ to be derived from the solution
M, N, D, t_i	known functions of λ
s, s_1	complex variables, $\mathrm{Re}(s) = \lambda$ for $\mathrm{Im}(s) = 0$
κ_j	$3 - 4\nu_j$
$\mu = \mu_2/\mu_1$	ratio of the shear moduli of the composite constituents
$\bar{\sigma}$, $\bar{v}$	Fourier transforms in the auxiliary problem
$\sigma^{\pm}$, $v^{\pm}$	factorized transforms
l	length parameter describing loading decay rate
b	$= 1/l$
$\breve{\mu}_j$	$\mu_j/(1 - \nu_j)$
q_0	loading amplitude
G	coefficient of the Wiener–Hopf problem
$G^{\pm}$, $G_1^{\pm}$, $G_2^{\pm}$, $R^{\pm}$	auxiliary functions analytical in the respective parts of the complex plane s
p	arbitrary coefficient multiplying the eigensolution
v_e	displacements of the crack faces in the eigenproblem
σ_e	stresses in front of the crack in the eigenproblem
$\sigma_\mathrm{e}^{\pm}$, $v_\mathrm{e}^{\pm}$	factorized transforms of the stresses displacements in the eigenproblem
K_n	stress intensity factor corresponding to the near field in the eigensolution
K_f	stress intensity factor corresponding to the remote field in the eigensolution
$\hat{K}$	ratio of the near stress intensity factor to the far one
$\hat{\sigma}_\mathrm{e}$	non-dimensional stresses in front of the crack
L, L_1	contours in the plane of complex variable s
ω	real parameter defining the location of the contour L_1
τ, ρ	real valued integration variables
f_σ	normalized stresses in front of the crack in the eigenproblem

considerations. A complete review of the crack problems in layered materials formulated in the framework of this assumption may be found in Hutchinson and Suo (1991).

Advances in modern technology have led to the employing of composite structures including very thin layers. The thickness of some kinds of coatings and adhesive layers may be of the order of microns. Therefore, in some cases the relation (1) does not hold and the above K concept fails, as it was noted by Xu et al. (1993). For sufficiently thin layers this phenomenon may take place even for composites with brittle constituents such as glass or epoxy which are usually examined in the framework of the stress intensity factor approach. In such a situation the fracture criterion formulation must be based on more precise considerations of the stress distribution not restricted by the first square root term. A corresponding analysis was carried out, for example, by Dyskin (1997) who considered the stipulation (1) in the context of rock mechanics.

Hence, for the cases when there is danger of violation of relation (1), deriving the size of the K-dominance zone R_K becomes a necessary step in the fracture analysis. Many researchers (Ma and Freund, 1986; Krishnaswamy et al., 1991; Huang and Gross, 1994) have investigated this issue analytically and numerically for static and dynamic problems of cracks in homogeneous materials. Additional references to experimental works may be found in Huang and Gross (1994).

Significantly, few studies are dedicated to the derivation of the K-dominance zone for cracks in inhomogeneous layered materials. A comprehensive numerical study of the stress field and the K-zone for the problem of a crack in the midplane of DCB adhesive fracture specimen is presented by Wang et al. (1978). Analysis of the limitations imposed by the extent of the K-dominance region on the use of the conventional stress intensity factor approach for the case of the mixed-mode delaminating beam specimen was carried out by Becker et al. (1997). The authors used finite elements to consider interface cracks for the bilayered and sandwich geometries.

When the crack is sufficiently long and the external loading on a body is smooth, the model of a semi-infinite crack with traction-free faces emerges, which provides universal information on the stress distribution in the crack tip vicinity. Xu et al. (1993) used this model in the problem of an anisotropic layer delaminating from a dissimilar half-space. The problem of a semi-infinite Mode III crack propagating at the interface between a layer and a semi-infinite substrate was treated by Ryvkin et al. (1995). In contrast to the previous problem, in the latter case the only loading is the decreasing remote stresses corresponding to the eigensolution. The analysis of the asymptotic properties of the eigensolution of this type in the sub-interface crack problems was carried out by Hutchinson et al. (1987).

The subject of the present paper is to obtain the complete eigensolution and derive the K-dominance zone for the problem of a semi-infinite crack in the midplane of an elastic layer sandwiched between two identical half-spaces (Fig. 1). The analysis of this problem by means of matched asymptotic expansions for the specific case of a layer considerably more compliant than the half-spaces was carried out by Banks-Sills and Salganik (1994). As was shown by theoretical studies (Fleck et al., 1991) and experimental investigations (Akisania and Fleck, 1992; Trantina, 1972), the midplane crack location is one of the possible propagation paths. Consequently, the universal information on the form of the stress distribution in this case, which will be provided by the eigensolution presented here, is of special interest.

In the following section the auxiliary problem of a loaded semi-infinite crack in a sandwich system is formulated and solved by means of the Wiener–Hopf method. In Section 3 the solution of the auxiliary problem is used to derive the eigensolution with traction-free crack faces by a limiting procedure. This

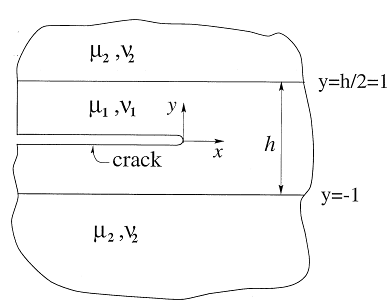

Fig. 1. Semi-infinite Mode I crack in a midplane of an elastic layer sandwiched between two identical half-spaces.

method for deriving the eigensolutions, suggested by Ryvkin et al. (1995), allows one to obtain the result in a closed form in contrast to the usually used dislocation approach which is based on a numerical solution of a singular integral equation (see, for example, Hutchinson et al., 1987; Thouless et al., 1987; Suo and Hutchinson, 1989; Fleck et al., 1991). At the same time it must be noted that the dislocation method is more general. The parametric study of the eigensolution presented in Section 4 yields the size of the K-dominance zone for any given parameter combination.

2. Auxiliary problem

Consider the plane deformation of a bi-material composite body consisting of an elastic layer sandwiched between two identical elastic half-spaces. A semi-infinite crack ($-\infty < x < 0, y = 0$) is located in the midplane of the layer (Fig. 1). Half of the layer thickness h is taken as unity, thus fixing the length scale for the analysis. The elastic properties of the composite constituents are defined by the shear moduli μ_j and Poisson's ratio ν_j (the value $j = 1$ corresponds to the material of the layer and $j = 2$ to the material of the half-spaces). The goal of the present paper is to obtain the specific eigensolution, namely, the solution corresponding to the traction-free crack faces and Mode I opening displacements in the crack tip vicinity. The possibility to obtain such a solution, i.e., to separate the fracture modes, is provided by the symmetry of the elastic domain.

In the auxiliary problem the stress state is generated by symmetric transverse opening tractions applied to the crack faces. Consequently, the deformation in the vicinity of the crack tip will be of the desired Mode I type. Due to the symmetry of the stress strain field, only the domain $y > 0$ need be considered. The boundary value problem for the layer $0 < y < 1$ ($j = 1$) bonded to the half-plane $1 < y < \infty$ ($j = 2$) is formulated as given below. The Lamé field equations with respect to the elastic displacements $u^{(j)}(x, y)$, $v^{(j)}(x, y)$ in the x and y directions respectively are

$$\nabla^2 u^{(j)} + \frac{1}{1 - 2\nu_j} \frac{\partial}{\partial x} \left(\frac{\partial u^{(j)}}{\partial x} + \frac{\partial v^{(j)}}{\partial y} \right) = 0 \tag{2}$$

$$\nabla^2 v^{(j)} + \frac{1}{1 - 2\nu_j} \frac{\partial}{\partial y} \left(\frac{\partial u^{(j)}}{\partial x} + \frac{\partial v^{(j)}}{\partial y} \right) = 0, \quad j = 1, 2. \tag{3}$$

At the interface between the layer and the half-space the components of the stress strain field satisfy the continuity conditions

$$u^{(2)}(x, 1) - u^{(1)}(x, 1) = 0, \tag{4}$$

$$v^{(2)}(x, 1) - v^{(1)}(x, 1) = 0, \tag{5}$$

$$\sigma_y^{(2)}(x, 1) - \sigma_y^{(1)}(x, 1) = 0, \tag{6}$$

$$\tau_{xy}^{(2)}(x, 1) - \tau_{xy}^{(1)}(x, 1) = 0, \tag{7}$$

and at the crack plane the mixed boundary conditions are

$$\tau_{xy}^{(1)}(x,0)=0, \quad -\infty<x<\infty, \tag{8}$$

$$v^{(1)}(x,0)=0, \quad 0<x<\infty, \tag{9}$$

$$\sigma_y^{(1)}(x,0)=q(x), \quad -\infty<x<0. \tag{10}$$

Here $q(x)<0$ is some known function defining the applied loading which is assumed to be localized in the vicinity of the crack tip. To complete the formulation of the boundary problem it is assumed that the local strain energy is bounded and that the stresses vanish at infinity:

$$\sigma_y^{(j)}(x,y),\ \sigma_x^{(j)}(x,y),\ \tau_{xy}^{(j)}(x,y)\rightarrow 0, \qquad x^2+y^2\rightarrow\infty. \tag{11}$$

The method for handling problem (2)–(11) using the Wiener–Hopf technique is straightforward. The displacements satisfying Eqs. (2) and (3) are represented in terms of the Fourier integrals,

$$f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\lambda,y)\mathrm{e}^{-i\lambda x}\mathrm{d}\lambda \tag{12}$$

It is convenient to represent them for the layer and the half-plane in the following slightly different forms (see, for example, Uflyand, 1968):

$$2\mu_1u^{(1)}(x,y)=\frac{i}{2\pi}\int_{-\infty}^{\infty}\left[A_1^{(1)}c+A_2^{(1)}s+\lambda y\left(A_3^{(1)}c+A_4^{(1)}s\right)\right]\mathrm{e}^{-i\lambda x}\,\mathrm{d}\lambda, \tag{13}$$

$$2\mu_1v^{(1)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-A_1^{(1)}s-A_2^{(1)}c+A_3^{(1)}\kappa_1c+A_4^{(1)}(k_1s-\lambda yc)\right]\mathrm{e}^{-i\lambda x}\,\mathrm{d}\lambda, \tag{14}$$

$$2\mu_2u^{(2)}(x,y)=\frac{-i}{2\pi}\int_{-\infty}^{\infty}\left(A_1^{(2)}+A_2^{(2)}\lambda y\right)\mathrm{e}^{-|\lambda|y-i\lambda x}\,\mathrm{d}\lambda, \tag{15}$$

$$2\mu_2v^{(2)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(A_1^{(2)}\frac{|\lambda|}{\lambda}+A_2^{(2)}(\kappa_2+|\lambda|y)\right)\mathrm{e}^{-|\lambda|y-i\lambda x}\,\mathrm{d}\lambda. \tag{16}$$

where $c=\cosh[\lambda(y-1)]$, $s=\sinh[\lambda(y-1)]$, $\kappa_j=3-4\nu_j$, $j=1,2$ and $A_r^{(j)}$, $r=1,2,3,4$ are eight unknown functions of λ to be determined from the boundary conditions. In the expressions for the half-plane (15), (16) two of these functions $A_3^{(2)}$ and $A_4^{(2)}$, being coefficients associated with the increasing exponentials, are equated to zero in accordance with the decreasing condition, *viZ* Eq. (11). Substituting Eqs. (13)–(16) into the five boundary conditions (4)–(8) by the use of the Hooke's law, one can express the remaining six functions and, consequently, the stress and displacements Fourier transforms in terms of one function, $A_3^{(1)}$. In particular, at the crack plane, we have

$$\bar{\sigma}(\lambda,0)=\frac{M(\lambda)}{D(\lambda)}A_3^{(1)}, \qquad \bar{v}(\lambda,0)=\frac{N(\lambda)}{D(\lambda)}A_3^{(1)} \tag{17}$$

where

$$M(\lambda)=\frac{\lambda^2}{2}\left[(\mu-1)t_1+8\mu(1-\nu_1)(\nu_2-\mu\nu_1)+t_2\cosh 2\lambda+2t_3\sinh 2|\lambda|\right], \tag{18}$$

$$N(\lambda) = \frac{(1-\nu_1)|\lambda|}{2\mu_1}\left[2\lambda(\mu-1)(\mu+\kappa_2) - 2t_3\cosh 2\lambda - t_2\sinh 2|\lambda|\right], \tag{19}$$

$$D(\lambda) = \left(\lambda^2 t_4 - |\lambda| t_3\right)\cosh\lambda + t_5\lambda\sinh\lambda \tag{20}$$

where

$$t_1 = 5\mu - 4\mu\nu_1 + 2\lambda^2(\mu+\kappa_2) + \kappa_2, \quad t_2 = \kappa_2 + \mu^2\kappa_1 + 2\mu(1-2\nu_1)(1-2\nu_2), \quad t_3 = 4\mu(1-\nu_1)(1-\nu_2),$$
$$t_4 = 2\mu(1-2\nu_2) + \mu^2 - \kappa_2, \quad t_5 = 2\mu\nu_1(1-\nu_2) - \mu^2(1-\nu_1) - \kappa_2,$$

and

$$\mu = \frac{\mu_2}{\mu_1} \tag{21}$$

is the shear moduli ratio of the layers.

Let us now consider the variable λ as being the real part of a complex variable s and split the transforms (17) into '+' and '−' functions which are regular in the upper $\mathrm{Im}(s) \geq 0$ and lower $\mathrm{Im}(s) \leq 0$ half-planes respectively. Namely, for $\mathrm{Im}(s) = 0$,

$$\bar{\sigma}_y(\lambda, 0) = \sigma^+(s) + \sigma^-(s), \tag{22}$$

$$\bar{v}(\lambda, 0) = v^+(s) + v^-(s). \tag{23}$$

Employing the mixed boundary conditions (9) and (10), one obtains the Wiener–Hopf equation

$$\sigma^+(s) = G(s)v^-(s) - \sigma^-(s), \quad s \in L, \tag{24}$$

where

$$\sigma^+(s) = \int_0^\infty \sigma_y(x, 0)e^{isx}\,dx \tag{25}$$

is the transform of the stresses in front of the crack, and where

$$v^-(s) = \int_{-\infty}^0 v(x, 0)e^{isx}\,dx \tag{26}$$

is the transform of the crack opening displacements. The contour L is located on the real axis $\mathrm{Im}(s) = 0$, and the coefficient of the problem, $G(s)$, is given by

$$G(s) = \frac{M(s)}{N(s)}. \tag{27}$$

Finally, the function $\sigma^-(s)$ is defined by the applied loading

$$\sigma^-(s) = \int_{-\infty}^0 q(x)e^{isx}\,dx. \tag{28}$$

As will be discussed later, the final result, i.e. the eigensolution, is independent of the specific form of the loading in the auxiliary problem; hence a decaying loading $q(x)$ may be chosen for reasons of convenience. Taking

$$q(x) = -q_0 \exp(x/l), \tag{29}$$

where l is a length parameter describing the loading decay rate, one obtains

$$\sigma^-(s) = -\frac{q_0}{b + is}, \qquad b = \frac{1}{l}. \tag{30}$$

The solution of Eq. (24) will be obtained as in Ryvkin et al. (1995). The behavior of the even function $G(\lambda)$ for small and large λ is defined by the elastic properties of the half-space and the layer respectively; namely,

$$G(\lambda) \sim -\breve{\mu}_j |\lambda|, \quad \breve{\mu}_j = \frac{\mu_j}{1 - \nu_j}, \tag{31}$$

where $j = 1$ for $\lambda \to \infty$ and $j = 2$ for $\lambda \to 0$. Consequently, the factorization of the function $G(\lambda)$

$$G(s) = \frac{G^+(s)}{G^-(s)}, \quad s \in L, \tag{32}$$

by the use of the Cauchy type integral (see Gakhov, 1966) is given by

$$G^{\pm}(s) = G_1^{\pm}(s) G_2^{\pm}(s) \tag{33}$$

$$G_1^+(s) = \sqrt{s + i0}, \qquad G_1^-(s) = -\frac{1}{\breve{\mu}_1 \sqrt{s - i0}}, \tag{34}$$

$$G_2^{\pm}(s) = \exp\left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln G_2(\tau)}{\tau - s}\, \mathrm{d}\tau\right], \tag{35}$$

where

$$G_2(\tau) = -\frac{G(\tau)}{\breve{\mu}_1 |\tau|}. \tag{36}$$

Here the expressions $s + i0$ and $s - i0$ in Eq. (34) are to be interpreted as that branch cut for the function $\sqrt{s}$ which is taken along the negative and positive imaginary axes, respectively. From Eqs. (18), (19), (27) and (31) it follows that the function $G_2(\tau)$ is positive, has a derivative, and tends to unity for $|\tau| \to \infty$ (The first property was verified numerically for all considered parameter combinations). Therefore the index of this function on the contour L is equal to zero and the representation equation (35) is correct.

Employing the above result makes it easy to factorize the inhomogeneous equation (24). Applying then the generalized Liouville theorem and using the assumption that the local strain energy is bounded, one obtains the solution of the inhomogeneous problem in the following form:

$$\sigma^+(s) = -R^+(s) G^+(s), \tag{37}$$

$$v^-(s) = R^-(s) G^-(s), \tag{38}$$

where

$$R^{+}(s)=\frac{iq_0}{s-ib}\left(\frac{1}{G^{+}(s)}-\frac{1}{G^{+}(ib)}\right) \tag{39}$$

$$R^{-}(s)=\frac{iq_0}{(s-ib)G^{+}(ib)}. \tag{40}$$

Hence, the formulas (13)–(22), (27), (30), (33)–(40) present the closed form solution for the auxiliary problem of a crack subjected to the exponentially decaying loading.

3. The eigensolution

In order to obtain the eigenproblem from the auxiliary one we will follow the way employed by Ryvkin et al. (1995), namely, we assume that

$$q_0=\frac{p}{\sqrt{l}}, \tag{41}$$

where p is an arbitrary constant having the dimensions of stress intensity factor and consider the limit as $l\to\infty$. Then, in accordance with Eq. (29), the crack faces become traction-free but the stress and strain fields do not degenerate to zero. This limiting distribution, proportional to p, which may be viewed as generated by the remote stresses applied at infinity corresponds to the eigenproblem. Consequently, the solution of the auxiliary problem yields the eigensolution; in particular,

$$\lim_{l\to\infty} v(x,y)=v_{\mathrm{e}}(x,y). \tag{42}$$

Hereafter, the subscript e denotes the eigensolution. The expressions for the displacements and stress transforms for the eigensolution are derived from Eqs. (37)–(40). Using Eq. (41) and taking the limit for $b\to 0$ (recall, that $b=1/l$) one obtains after some manipulations

$$\sigma_{\mathrm{e}}^{+}(s)=\frac{p}{G_2^{+}(0)}\exp\left(i\frac{\pi}{4}\right)\frac{G_2^{+}(s)}{\sqrt{s+i0}}, \tag{43}$$

$$v_{\mathrm{e}}^{-}(s)=-\frac{p}{\breve{\mu}_1 G_2^{+}(0)}\exp\left(i\frac{\pi}{4}\right)\frac{G_2^{-}(s)}{(s-i0)^{3/2}}, \tag{44}$$

where the functions $G_2^{\pm}(s)$ are defined by Eq. (35). Since for the eigenproblem $\sigma_{\mathrm{e}}^{-}(s)=v_{\mathrm{e}}^{+}(s)=0$, the stress distribution in front of the crack tip and the crack opening displacements are expressed by the inverse Fourier transform integrals as following

$$\left\{\begin{array}{c}\sigma_{\mathrm{e}}(x)\\ v_{\mathrm{e}}(x)\end{array}\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\begin{array}{c}\sigma_{\mathrm{e}}^{+}(s)\\ v_{\mathrm{e}}^{-}(s)\end{array}\right\}\exp(-isx)\,\mathrm{d}s. \tag{45}$$

The definitions $\sigma_{\mathrm{e}}(x)\equiv\sigma_{y\mathrm{e}}(x,0)$ and $v_{\mathrm{e}}(x)\equiv v(x,0)$ have been adopted hereafter for brevity. Note, that from Eqs. (43)–(45) it follows that the eigensolution includes an arbitrary multiplier p defining its amplitude. Another important remark is that the layer thickness is the only length parameter and, consequently, the only possible length scale for this solution.

The near and far field solutions for the eigenproblem can be derived from result obtained by the usual methods. The asymptotes for the displacements and stresses for $x\to 0$ and $x\to\infty$ are defined by the asymptotic behavior of their transforms (43), (44) for large and small $|s|$, respectively. For $|s| \to \infty$ in the respective half-planes we have

$$\sigma_e^+(s) \sim \frac{p}{G_2^+(0)}(-is)^{-1/2} \tag{46}$$

and

$$v_e^-(s) \sim \frac{p}{\breve{\mu}_1 G_2^+(0)}(is)^{-3/2}, \tag{47}$$

Consequently, the near asymptotes are found to be

$$\sigma(x) \sim p\sqrt{\frac{\breve{\mu}_1}{\pi\breve{\mu}_2}}x^{-1/2} \quad \text{for } x \to +0 \tag{48}$$

and

$$v(x) \sim \frac{2p}{\sqrt{\breve{\mu}_1\breve{\mu}_2\pi}}(-x)^{1/2} \quad \text{for } x \to -0. \tag{49}$$

The far field asymptotes in the eigenproblem, in contrast to the auxiliary one, are also defined by the same square root functions multiplied by the different coefficients. In fact, based on the asymptotic behavior of the transforms for small s,

$$\sigma_e^+(s) \sim p(-is)^{-1/2}, \quad \mathrm{Im}(s) \to +0, \tag{50}$$

$$v_e^-(s) \sim \frac{p}{\breve{\mu}_2}(is)^{-3/2}, \quad \mathrm{Im}(s) \to -0, \tag{51}$$

one obtains

$$\sigma_e(x) \sim \frac{p}{\sqrt{\pi}}x^{-1/2}, \quad x \to +\infty, \tag{52}$$

and

$$v_e(x) \sim \frac{2p}{\sqrt{\breve{\mu}_2\pi}}(-x)^{1/2}, \quad x \to -\infty, \tag{53}$$

The near and far stress intensity factors K_n and K_f are defined by the relations

$$K_n = \lim_{x\to 0}\sqrt{2\pi x}\sigma_e(x), \tag{54}$$

$$K_f = \lim_{x\to\infty}\sqrt{2\pi x}\sigma_e(x), \tag{55}$$

and the asymptotes (48) and (52) yield

$$K_{\mathrm{n}} = p\sqrt{2\frac{\breve{\mu}_1}{\breve{\mu}_2}}, \qquad K_{\mathrm{f}} = p\sqrt{2}. \tag{56}$$

Since, as noted, the eigensolution includes an arbitrary multiplier, the information on the stress distribution in front of the crack is provided not by the magnitudes of the stress intensity factors but by their ratio

$$\hat{K} = \frac{K_{\mathrm{n}}}{K_{\mathrm{f}}} = \sqrt{\frac{\breve{\mu}_1}{\breve{\mu}_2}}. \tag{57}$$

For the problem of a sufficiently long uniformly loaded crack of length l in the midplane of a sandwich composite, this ratio clearly will express the relation between the near field ($x \ll h$) and the far field ($h \ll x \ll l$) stress intensity factors. An alternative more simple method to obtain Eq. (57) is to equate the energy release rates calculated in terms of the near and remote fields as was done Fleck et al. (1991). However, it should be noted that this method yields only a relation between the asymptotes of the eigensolution but not lead to a determination of the complete stress distribution as obtained in the present work. Another way to derive Eq. (57), based on the asymptotic approach, was employed by Banks-Sills and Salganik (1994). The existence of the two stress intensity factors depending upon the scale of consideration has been clearly illustrated by Erdogan and Ozturk (1992).

The method of deriving the eigensolution used in the present paper may raise a question on its dependence upon the specific loading adopted in the auxiliary problem. The absence of this dependence follows from the uniqueness of the Mode I eigensolution for the considered layered body with a semi-infinite crack. The formal proof of the uniqueness is based on the standard use of the Kirchhoff theorem in conjunction with some reasonable assumptions regarding the behavior of the solution near the crack tip and at infinity.

4. Numerical results

In this section the stress distribution in front of the crack in the eigensolution is investigated, and the size of the K-dominance zone for the near stress intensity factor is determined. To study an eigensolution one must choose the reference point. In the case considered, as in Ryvkin et al. (1995), it is plausible to assume that the far field characterized by the stress intensity factor K_{f} (see Eq. (55)) is fixed. Consequently, noting that for the formulation considered in the present problem with dimensionless length units, stress intensity factor has dimensions of stress, the non-dimensional stress is defined as follows:

$$\hat{\sigma}_{\mathrm{e}}(x) = \frac{\sigma_{\mathrm{e}}(x)}{K_{\mathrm{f}}}. \tag{58}$$

The evaluation of the function $\hat{\sigma}_{\mathrm{e}}(x)$ which is expressed, in accordance with Eqs. (35), (43) and (45), by a double integral with infinite limits poses some numerical problems. The inner Cauchy type integral equation (43) converges exponentially, since, as follows from Eqs. (18), (19), (27) and (36),

$$G_2(\tau) \sim 1 + O\big[\exp(-2|\tau|)\big] \quad \text{for } |\tau| \to \infty, \tag{59}$$

where O represents the order of magnitude. On the other hand, in the outer integral, for the large $|s|$, the integrand behaves as $O[s^{-1/2}\exp(isx)]$ and the calculations present severe difficulties. In order to improve the convergence of the outer integral the integration path is deformed from the real axis to the lower half-plane $\mathrm{Im}(s) \leq 0$ using the Cauchy theorem. Then employing the symmetric properties of the integrand and applying the homogeneous Wiener–Hopf equation

$$\sigma_e^+(s) = G(s)v_e^-(s), \quad s \in L, \tag{60}$$

to carry out analytical continuation through the real axis, one obtains

$$\hat{\sigma}_e(x) = \frac{1}{\pi}\mathrm{Re}\left[\int_{L_1} G(s)v_e^-(s)\exp(-isx)\,\mathrm{d}s\right], \quad s \in L_1. \tag{61}$$

Here the contour L_1 is a ray $s = \rho\exp(-i\omega)$, $0 \leq \rho \leq \infty$. The ray direction, defined by the value of $\omega > 0$, is chosen using the same reasoning as in Ryvkin et al. (1995). Consequently, for a large $|s| = \rho$, the integrand of the outer integral in Eq. (61) decays exponentially as $O[\rho^{-1/2}\exp(-\rho x \sin\omega)]$ making the numerical procedure of the stress evaluation considerably more effective. For all the material parameter combinations considered it was found that in order to get a precision to three significant figures, it is possible to replace the infinite integration regions in the inner and outer integrals by the intervals $-6 < \tau < 6$ and $0 < \rho < 100$, respectively.

The non-dimensional stress $\hat{\sigma}_e$ is seen to be a function of the four non-dimensional parameters, namely

$$\hat{\sigma}_e = \hat{\sigma}_e(\mu, x, \nu_1, \nu_2). \tag{62}$$

Recall that $\mu = \mu_2/\mu_1$ and all the length quantities are normalized by equating half of the cracked layer thickness h to unity. The graph of the function $\hat{\sigma}_e(x)$ for $\mu = 10$, $\nu_1 = 0.35$ and $\nu_2 = 0.3$ is exhibited in Fig. 2(a) and (b) in different scales. The plot of the eigensolution is shown as the solid line. The different dashed lines indicate the near and far square root asymptotes. In accordance with Eqs. (56)–(58) they were found to be

$$\hat{\sigma}_e(x) \sim \frac{\hat{K}}{\sqrt{2\pi x}}, \quad \text{for } x \to 0, \tag{63}$$

and

$$\hat{\sigma}_e(x) \sim \frac{1}{\sqrt{2\pi x}}, \quad \text{for } x \to \infty. \tag{64}$$

In the case considered, $\mu = 10$, the layer is more compliant than the half-spaces. From Eqs. (31) and (57) it follows that for this case $\hat{K} < 1$ and the graph of the far asymptote is located above the graph for the near one in agreement with the known elastic stress shielding effect. In accordance with this effect, observed for the different sandwich systems considered by Wang et al. (1978) and Fleck et al. (1991), for the given remote loading the stresses in the vicinity of the crack tip in a compliant layer are diminished in comparison to the case of a crack in a homogeneous material. In Hutchinson and Suo (1991) it was noted that if flaws controlling the strength of the composite are much smaller than the layer thickness the shielding effect is reduced. In the context of the present study it must be added that for the second limiting case of a large ratio of the crack length to the layer thickness, the shielding effect may also be less pronounced. This situation appears when the layer thickness is not small with respect to the fracture process zone. Consequently, the stresses in the crack tip vicinity involved in the fracture criterion formulation may be essentially higher than predicted by the near asymptote (see Fig. 2(a)).

Fig. 2(b) shows that in contrast to the near asymptote the far one is approached from above, i.e., the region where the stress is diminished with respect to the case of a crack in a homogeneous body is followed by the region where it is enlarged. This phenomenon is backed up by the relation

$$\int_0^\infty \left[\hat{\sigma}_e(x) - \frac{1}{\sqrt{2\pi x}} \right] dx = 0 \tag{65}$$

which is obtained from the equilibrium condition of the domain $y > 0$. The region of switching from the near to the far asymptote is characterized by relatively small stress variations (see Fig. 2(a)). The results exhibited in Fig. 3 show that when the rigidity of the half-spaces increases the stress in this region tends to attain a constant value. (The limiting distribution corresponds to the problem considered by Knauss (1966) of a cracked layer with the clamped boundaries which are displaced normal to the crack.) This feature of the stress distribution in the switching region was employed in the asymptotic analysis presented by Banks-Sills and Salganik (1994).

The dependence of the stress distribution upon the Poisson ratio of the layer is illustrated in Fig. 4. Three values of $\nu_1 = 0.2, 0.35, 0.5$ for the composite with $\mu = 30$ and $\nu_2 = 0.3$ are examined. As expected, the Poisson ratio scarcely affects the stress field. Only for the limiting case of an incompressible material $\nu_1 = 0.5$ is a significant difference from the cases with moderate values of ν_1

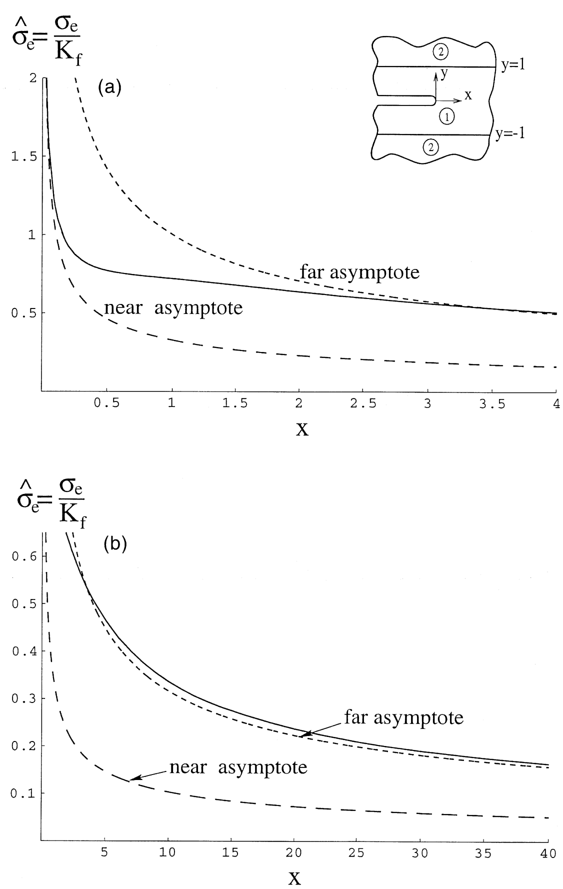

Fig. 2. Non-dimensional stresses in front of the crack $\hat{\sigma}_e(x)$ for the eigensolution using fine (a) and coarse (b) scales. The near and the far square root asymptotes are shown by dashed lines. The composite with shear moduli ratio $\mu = \mu_2/\mu_1 = 10$ and Poisson ratios $\nu_1 = 0.35$, $\nu_2 = 0.3$ is considered.

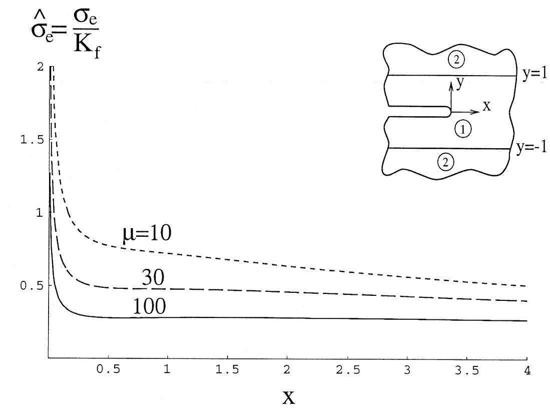

Fig. 3. Influence of the shear moduli ratio $\mu = \mu_2/\mu_1$ of the composite constituents on the stress distribution in front of the crack in the eigensolution for typical values $\nu_1 = 0.35$ and $\nu_2 = 0.3$.

observed. For this case, the general form of the stress curve with a weak minimum is in agreement with the distribution obtained by Knauss (1966).

For further analysis of the stress distribution for small and large x, it is convenient to eliminate the square root singularity. Defining the normalized quantity as

$$f_\sigma(x) = \sqrt{2\pi x}\,\hat{\sigma}_e(x) \tag{66}$$

and introducing a new variable $s_1 = s/x$, one obtains a convenient formula

$$f_\sigma(x) = \sqrt{\frac{\breve{\mu}_1}{\pi\breve{\mu}_2}}\mathrm{Re}\left[\int_{L_1} G_2^-(s_1/x)G_2(s_1/x)\frac{\exp[i(\pi/4 - s_1)]}{\sqrt{s_1 - i0}}\,\mathrm{d}s_1\right]. \tag{67}$$

In Fig. 5(a) and (b) the normalized stresses distributions for the elastic shear moduli ratios $\mu = 2$, 5, 30,

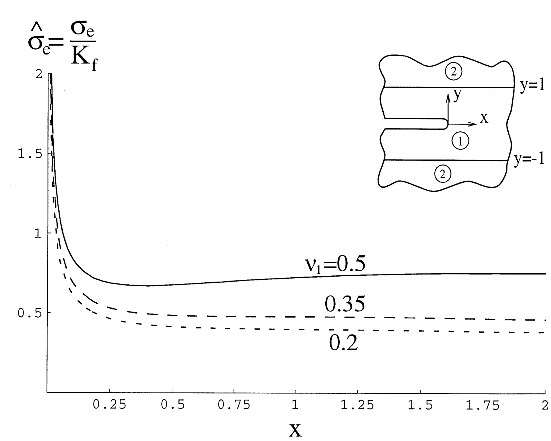

Fig. 4. Influence of the cracked layer Poisson ratio on the stress distribution in front of the crack for a composite with $\mu = 30$ and $\nu_2 = 0.3$.

100 are exhibited. The Poisson ratios are taken as $\nu_1 = 0.35$ and $\nu_2 = 0.3$ for all cases. The near and far asymptotes (63) and (64) correspond now to the horizontal straight lines.

For $x \to \infty$, all the curves approach the far asymptote $f_\sigma(x) = 1$ corresponding to the degenerate case of identical materials of the layer and the half-spaces. The region of validity of this asymptote is seen to decrease with increasing mismatch of the elastic properties of the sandwich constituents. At the same time it should be noted that the specific form of the curves $f_\sigma(x)$ with a weak maximum may cause this decrease to become non-monotonic for some ranges of μ.

The behavior of the stresses in the crack tip vicinity defining the size of the K-dominance zone, i.e., the region of validity of the near asymptote (63), is of particular interest. From Eqs. (64), (57), (58) and (66) it follows that

$$\lim_{x \to 0} f_\sigma(x) = \sqrt{\frac{\breve{\mu}_1}{\breve{\mu}_2}}. \tag{68}$$

Hence, in accordance with Eq. (31), the level of stresses in the crack tip vicinity is strongly dependent upon the shear moduli ratio μ. At the same time, from Fig. 5(b) it is seen that the influence of this parameter on the derivative of the function $f_\sigma(x)$ near the point $x = 0$ is insignificant. Consequently, the

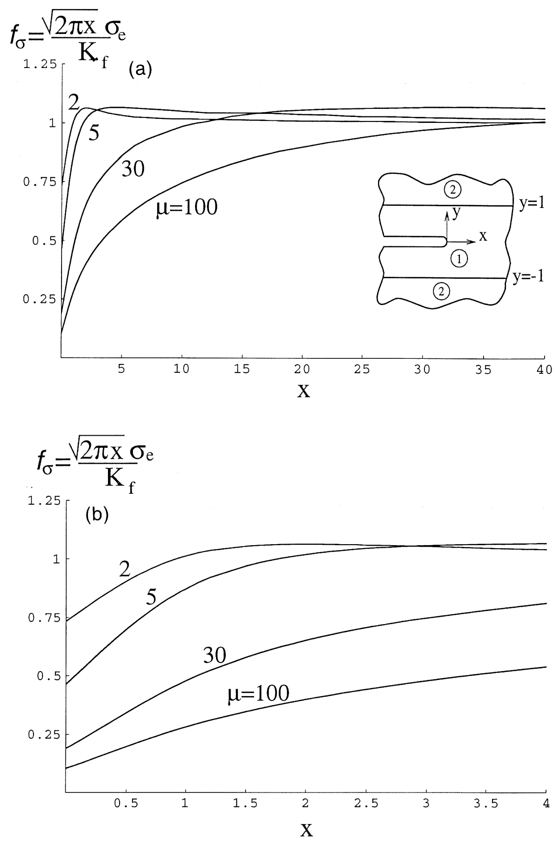

Fig. 5. Normalized stress distribution $f_\sigma(x) = \sqrt{2\pi x}\hat{\sigma}_e(x)$ in the eigensolution using coarse (a) and fine (b) scales showing the influence of the shear moduli ratio $\mu = \mu_2/\mu_1$ for a composite with $\nu_1 = 0.35$, and $\nu_2 = 0.3$.

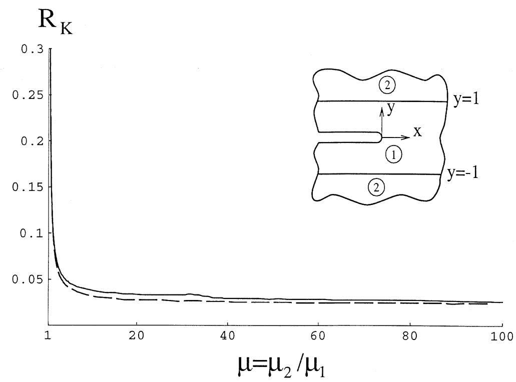

Fig. 6. Non-dimensional size of the K-dominance zone R_K vs. shear moduli ratio of the composite constituents for two Poisson ratio combinations: $\nu_1 = 0.35$, $\nu_2 = 0.3$ (solid line) and $\nu_1 = 0.45$, $\nu_2 = 0.3$ (dashed line).

distance from the crack tip at which a specified stress deviation from the near asymptote occurs is not sensitive to μ. The same effect was observed by Wang et al. (1978) based on a finite element analysis of a double-cantilever sandwich specimen.

The graphical results illustrating the dependence of the size of the K-dominance zone R_K upon the shear moduli ratio are exhibited in Fig. 6 (Recall, that all the length quantities in the problem are measured in units of half of the cracked layer thickness). The value R_K for each material parameter combination was derived numerically from the condition of 5% relative discrepancy between the stress $\hat{\sigma}(x)$ and its near asymptote. In terms of the function f_σ, this condition is given as

$$f_\sigma(R_K) = 1.05 f_\sigma(0) \tag{69}$$

The monotonic shrinking of the K-dominance zone with the increasing dissimilarity of the materials is observed. A steep decreasing for $1 < \mu < 10$ is followed by the region where R_K is approximately constant and equal to 0.03. A comparison of the two curves for different sets of Poisson ratio indicates that the influence of these parameters on the size of the K-dominance zone is limited. Consequently, the result obtained, namely that the size of the K-dominance zone in front of the crack is equal to 1.5% of the cracked layer thickness, is valid for any materials combination if their properties are not too close to each other. This result is also in a good agreement with the data reported by Wang et al. (1978). For the sandwich composite considered in that work the corresponding size, calculated on the basis of a 10% relative stress variation, was found to be 2.5%.

5. Summary

The stress state in the problem of a semi-infinite Mode I midplane crack in a layer sandwiched between two identical half-spaces is investigated by means of the corresponding eigensolution. The eigensolution is obtained in a closed form convenient for calculations. A complete stress distribution having the near and the remote square root asymptotes is derived. Since the cracked layer thickness is the only length parameter of the eigenproblem, the solution obtained provides a universal information on the stress distribution near the tip of a sufficiently long crack in a sandwich composite.

The numerical evaluation of the solution for the case when the layer is more compliant than the half-spaces led to a determination of the exact size R_K of the K-dominance zone. It appears that if the shear moduli ratio of the composite constituents is more than 10, this size becomes roughly the same for any set of materials. Calculated for the 5% relative stress variation, R_K is found to be about 1.5% of the layer thickness. Consequently, in the case of thin layers the K-concept may become invalid even for brittle adhesives. In fact, in accordance with the results obtained here, the actual size of the K-dominance zone for a $50\mu\kappa$ thickness epoxy adhesive is $0.75\mu\kappa$, which is less than the magnitude of the process zone $R_p = 5\mu\kappa$ given in the literature (see Fleck et al., 1991). Incorporating the dynamic effects in the analysis will lead to further reduction of the K-dominance zone and restrictions on employing the conventional K-concept.

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